

An elementary approach to Gaussian multiplicative chaos

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Abstract

A completely elementary and self-contained proof of convergence of Gaussian multiplicative chaos is given. The argument shows further that the limiting random measure is nontrivial in the entire subcritical phase ($\gamma < \sqrt{2d}$) and that the limit is universal (i.e., the limiting measure is independent of the regularisation of the underlying field).

Keywords: Gaussian multiplicative chaos, Gaussian free field, thick points, Liouville quantum gravity.

1 Introduction

Gaussian multiplicative chaos is a theory initiated by Kahane [9], whose goal (in a slightly updated language) is the definition and study of random measures in domains of the form

$$\mu(dx) = e^{\gamma h(x) - \frac{1}{2}\gamma^2 \mathbb{E}(h(x)^2)} \sigma(dx) \quad (1.1)$$

where h is a centred Gaussian generalised random field subject to certain assumptions, $\gamma > 0$, and σ is a given reference measure. Since h is not defined pointwise but exists only as a distribution, it is not clear what meaning to give to (1.1) a priori. In fact, some regularisation of the field and a suitable renormalisation have to be performed in order to construct μ . The theory has generated considerable renewed interest notably because of its connection with two-dimensional Liouville Quantum Gravity and the KPZ relations. This is the particular case of the theory when $d = 2$ and h is the massless Gaussian free field or GFF (see [13]) with appropriate boundary conditions. The paper by Duplantier and Sheffield [3] constructed the volume measure μ in this particular case using arguments restricted to the case of the GFF such as the domain Markov property and obtained a version of the KPZ relations. Simultaneously, Rhodes and Vargas, [10], among other things, showed that Gaussian multiplicative chaos can be used directly to construct the same object. They also gave a simpler and more general proof of a slightly stronger version of the KPZ relation. We refer the reader to [5] for an excellent survey of relatively recent mathematical developments and general background in this area.

Kahane's original work assumes that the covariance kernel K of h is σ -positive, meaning that $K(x, y)$ may be written as the pointwise sum $K(x, y) = \sum_k K_k(x, y)$ where the summands are nonnegative symmetric definite continuous functions and, crucially, are pointwise nonnegative. Under this assumption (which is somewhat restrictive as it is hard to check in practice), he was

*Supported in part by EPSRC grants EP/L018896/1 and EP/I03372X/1

able to show that a truncation of h associated with the σ -positive decomposition of K gives rise to a well-defined measure μ as in (1.1) and characterised the values of γ for which it is nontrivial for a given reference measure σ . He also studied fine properties of the resulting random measure μ and showed that its law does not depend on the decomposition of the σ -positive kernel K into positive summands.

Much more recently, Robert and Vargas [11] (motivated by applications to three-dimensional turbulence) obtained a significant generalisation of this theory. They were able to show that, without assuming σ -positivity, regularising the field with a general mollifier function θ gives rise to a sequence of measures μ_ε such that $\mu_\varepsilon(S)$ converges in law and the law of the limit does not depend on the regularising function θ . Even more recently, Shamov [12] showed in a very general setting that convergence holds in probability and the limit does not depend on the regularisation. In particular the measure μ is measurable with respect to h . (Conversely, h is measurable with respect to μ , at least in the case of the two-dimensional GFF by a result of [2]). This was also the subject of a recent preprint by Junnila and Saksman [8] whose results, remarkably, cover the critical case.

The purpose of this short note is to provide an elementary and completely self-contained proof of Kahane's theory together with some of the important developments above. Eventually we are able to reprove the convergence in probability (and in L^1) result and show nontriviality in the entire subcritical phase ($\gamma < \sqrt{2d}$), together with the universality result showing uniqueness of the limit (independence with respect to the regularisation function θ). While the setup is slightly less general than Shamov [12], we feel that the result and its proof are nevertheless interesting because of the completely elementary nature of the arguments, and the fact that they cover the most interesting cases without significant assumptions on the covariance kernel K (in particular, no σ -positivity assumption is made).

Assumptions: Let h be a Gaussian generalised function on a domain $D \subset \mathbb{R}^k$. Let σ be a Radon measure on D of dimension at least $d \leq k$, i.e.,

$$\iint \frac{1}{|x-y|^{d-\varepsilon}} \sigma(dx) \sigma(dy) < \infty \quad (1.2)$$

for all $\varepsilon > 0$. We assume that the covariance of h satisfies

$$\text{Cov}(h(x), h(y)) = K(x, y) = \log(|x-y|^{-1}) + g(x, y) \quad (1.3)$$

where g is smooth over $\bar{D} \times \bar{D}$, meaning that for any smooth test function f supported in D ,

$$(h, f) \text{ is a centered Gaussian random variable with variance } \iint f(x) K(x, y) f(y) dx dy.$$

Note that this covers the case of a Gaussian free field in two dimensions with say Dirichlet boundary conditions (but also the case of free boundary conditions, by changing if necessary γ into 2γ).

Assume without loss of generality that D contains the ball of radius 10 around the origin and let S be the unit cube $(0, 1)^k$. Let $\theta : \mathbb{R}^k \rightarrow [0, 1]$ be a fixed smooth function of compact support and such that $\int \theta(x) dx = 1$. Set $\theta_\varepsilon(x)$ to be an approximation of identity based on θ , i.e. $\theta_\varepsilon(x) = \varepsilon^{-k} \theta(x/\varepsilon)$. We then consider an ε -regularisation of the field h by setting

$$h_\varepsilon(x) = h \star \theta_\varepsilon(x). \quad (1.4)$$

Remark 1.1. In certain situations one may wish to relax the assumption of smoothness on θ . In fact this is not used at any point in the proof, except to guarantee that $X \star \theta_\varepsilon$ is well defined as a random variable. If this fact is known for other reasons (e.g. if h is the two-dimensional GFF and θ is the uniform distribution on the unit circle, in which case h_ε is simply the usual circle average), then the theorems and arguments below go through without any change.

Let

$$\mu_\varepsilon(S) = I_\varepsilon = \int_S e^{\gamma h_\varepsilon(z) - \frac{\gamma^2}{2} \mathbb{E}(h_\varepsilon(z)^2)} \sigma(dz).$$

In the case of 2d Liouville quantum gravity, the natural choice for σ is often $\sigma(dz) = R(z, D)^{\gamma^2/2}$ where $R(z, D)$ denotes the conformal radius of the point z in D . However, the case where σ is the occupation measure of an (independent) complex Brownian motion is also of interest as it can be used for defining the Liouville Brownian motion [6, 1], the canonical diffusion process in Liouville quantum gravity. This is an example where σ is singular with respect to Lebesgue measure but is nevertheless of dimension two in the sense of (1.2).

Theorem 1.2. *Assume that $\gamma < \sqrt{2d}$. Then $\mu_\varepsilon(S)$ converges in probability and in L^1 to a limit $\mu(S)$. Furthermore $\mu(S)$ does not depend on the choice of the regularising kernel θ subject to the above assumptions.*

Acknowledgements: The idea for this work emerged while preparing a presentation for a reading group at the Newton institute during the semester on *Random Geometry*. I thank the participants of the reading group for their questions. I am particularly grateful to Juhan Aru and Stéphane Benoît for some useful discussions and comments on a preliminary draft. Thanks also to Ofer Zeitouni for some comments on the Karhunen–Loeve expansion of Gaussian fields.

2 Main idea

It is well known and relatively easy to see that for γ sufficiently small (namely $\gamma < \sqrt{d}$), the multiplicative measure μ_ε are uniformly integrable: indeed the quantity $\mu_\varepsilon(S)$ is then bounded in L^2 , hence any limit must be nontrivial.

Therefore difficulties mainly arise in the phase where $\gamma \in [\sqrt{d}, \sqrt{2d})$. (In Liouville Quantum Gravity, this is the phase of principal interest as this is precisely the measures which are thought to arise as scaling limits of FK-weighted planar maps). The main idea for this work is the following very elementary observation. Any limiting measure μ must be supported on the so-called γ -thick points of the field h : that is, on points x such that

$$\lim_{\varepsilon \rightarrow 0} \frac{h_\varepsilon(x)}{\log(1/\varepsilon)} = \gamma. \quad (2.1)$$

Such points were studied in detail in the case of the two-dimensional Gaussian Free Field by [7] but a related notion was already apparent in the early work of Kahane [9] who pointed to its importance. That any limiting measure would have to be supported by γ -thick points is apparent from the definition of μ_ε and Girsanov’s lemma, which implies that when biasing the law of the field by a factor proportional to $e^{\gamma h_\varepsilon(x)}$, the mean value of $h_\varepsilon(x)$ is shifted from 0 to $\gamma \text{Var}(h_\varepsilon(x)) = \gamma \log(1/\varepsilon) + O(1)$.

Therefore, one can pick $\alpha > \gamma$ and call a point x bad if its thickness is greater than α . We then consider the normalised measure $e^{\gamma h_\varepsilon(x)} dx$, but *restricted to good points*. As it turns out, the L^1 contribution of bad points is easily shown to be negligible (essentially by the above Girsanov observation), while the remaining part is shown to remain bounded in L^2 . We will see that a suitable definition of *good points* (i.e., not too thick) allows one to make the relevant L^2 computation very simple.

Convergence is then shown to be a consequence of the L^2 boundedness (roughly, the good part is a Cauchy sequence in L^2). Uniqueness comes from the fact that once uniform integrability is established it also follows for another martingale approximation of the measure arising from the Karhunen-Loeve expansion of h . We then show that the two measures must agree, thereby deducing uniqueness.

3 Uniform Integrability

The goal of this section will be to prove:

Proposition 3.1. *I_ε is uniformly integrable.*

Proof. Let $\alpha > 0$ be fixed (it will be chosen $> \gamma$ and very close to γ soon). We define a good event

$$G_\varepsilon^\alpha(x) = \{h_{\varepsilon'}(x) \leq \alpha \log(1/\varepsilon') \text{ for all } \varepsilon' \in [\varepsilon, \varepsilon_0]\}$$

with $\varepsilon_0 \leq 1$ for instance. This is the good event that the point x is never too thick up to scale ε .

Further let $\bar{h}_\varepsilon(x) = \gamma h_\varepsilon(x) - (\gamma^2/2)\mathbb{E}(h_\varepsilon(x)^2)$ to ease notations.

Lemma 3.2 (Ordinary points are not thick). *For any $\alpha > 0$, we have that uniformly over $x \in S$, $\mathbb{P}(G_\varepsilon^\alpha(x)) \geq 1 - p(\varepsilon_0)$ where the function p may depend on α and for a fixed $\alpha > \gamma$, $p(\varepsilon_0) \rightarrow 0$ as $\varepsilon_0 \rightarrow 0$.*

Remark 3.3. The proof of this lemma is trivial in the case of the two-dimensional GFF (in particular no general machinery about Gaussian processes is needed in this case).

Proof. Set $X_t = h_\varepsilon(x)$ for $\varepsilon = -\log t$. Then a direct computation from (1.3) shows that $\text{Cov}(X_s, X_t)$ is a smooth function of s and t , and moreover,

$$|\text{Cov}(X_s, X_t) - s \wedge t| \leq O(1), \quad (3.1)$$

where the implicit constant is uniform. In particular $\text{Var}(X_t) = t + O(1)$.

Note that for each $k \geq 1$, $\mathbb{P}(X_k \geq \alpha k/2) \leq e^{-\alpha^2 k^2 / (8 \text{Var}(X_k))}$ which decays exponentially in k by the above. Furthermore, on each interval $[k, k+1]$, uniformly in $k \geq 0$, the process $(X_{k+t} - X_k; t \in [0, 1])$ is Gaussian, with a smooth covariance function (and derivative uniformly bounded in k). Fernique's theorem (in fact, a rather simple case of that theorem) implies that $\mathbb{E}(\sup_{t \in [0, 1]} (X_{t+k} - X_k)) \leq C$ uniformly in k (see e.g. Theorem 4 in [14]). In particular, by Borell's inequality (Theorem 2 in [14]), $\mathbb{P}(\sup_{t \in [0, 1]} (X_{t+k} - X_k) \geq \alpha k/2)$ decays also exponentially in k^2 . Hence

$$\mathbb{P}(\exists t \geq k_0 : |X_t| \geq \alpha t) \leq \sum_{k=k_0}^{\infty} \mathbb{P}(X_k \geq \alpha k/2) + 2\mathbb{P}(\sup_{t \in [0, 1]} (X_{t+k} - X_k) \geq \alpha k/2)$$

We call $p(\varepsilon_0)$ to be the right hand side of the above for $k_0 = \lceil -\log(1/\varepsilon_0) \rceil$ which can be made arbitrarily small by picking ε_0 small enough. This proves the lemma. \square

Lemma 3.4 (Liouville points are no more than γ -thick). *For $\alpha > \gamma$ we have*

$$\mathbb{E}(e^{\bar{h}_\varepsilon(x)} \mathbf{1}_{G_\varepsilon^\alpha(x)}) \geq 1 - p(\varepsilon_0).$$

Proof. Note that

$$\mathbb{E}(e^{\bar{h}_\varepsilon(x)} \mathbf{1}_{\{G_\varepsilon^\alpha(x)\}}) = \tilde{\mathbb{P}}(G_\varepsilon^\alpha(x)), \text{ where } \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{\bar{h}_\varepsilon(x)}.$$

By Girsanov's lemma, under $\tilde{\mathbb{P}}$, the process X_s has the same covariance structure as under \mathbb{P} and its mean is now $\gamma \text{Cov}(X_s, X_t) = \gamma s + O(1)$ for $s \leq t$. Hence

$$\tilde{\mathbb{P}}(G_\varepsilon^\alpha(x)) \geq \mathbb{P}(G_\varepsilon^{\alpha-\gamma}(x)) \geq 1 - p(\varepsilon_0)$$

by Lemma 3.2 since $\alpha > \gamma$. □

We therefore see that points which are more than γ -thick do not contribute significantly to I_ε in expectation and can therefore be safely removed. We therefore fix $\alpha > \gamma$ and introduce:

$$J_\varepsilon = \int_S e^{\bar{h}_\varepsilon(z)} \mathbf{1}_{\{G_\varepsilon(z)\}} \sigma(dz) \quad (3.2)$$

with $G_\varepsilon(x) = G_\varepsilon^\alpha(x)$. We will show that J_ε is uniformly integrable from which the result follows.

Lemma 3.5. *J_ε is bounded in L^2 and hence uniformly integrable.*

Proof. By Fubini's theorem,

$$\begin{aligned} \mathbb{E}(J_\varepsilon^2) &= \int_{S \times S} \mathbb{E}(e^{\bar{h}_\varepsilon(x) + \bar{h}_\varepsilon(y)} \mathbf{1}_{\{G_\varepsilon(x), G_\varepsilon(y)\}}) \sigma(dx) \sigma(dy) \\ &= \int_{S \times S} e^{\gamma^2 \text{Cov}(h_\varepsilon(x), h_\varepsilon(y))} \tilde{\mathbb{P}}(G_\varepsilon(x), G_\varepsilon(y)) \sigma(dx) \sigma(dy) \end{aligned}$$

where $\tilde{\mathbb{P}}$ is a new probability measure obtained by the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\bar{h}_\varepsilon(x) + \bar{h}_\varepsilon(y)}}{\mathbb{E}(e^{\bar{h}_\varepsilon(x) + \bar{h}_\varepsilon(y)})}.$$

Observe that for $|x - y| \leq 3\varepsilon$ say, by Cauchy-Schwarz and (3.1), $\text{Cov}(h_\varepsilon(x), h_\varepsilon(y)) \leq \log(1/\varepsilon) + g(x, x) + o(1)$. On the other hand, for $|x - y| \geq 3\varepsilon$,

$$\begin{aligned} \text{Cov}(h_\varepsilon(x), h_\varepsilon(y)) &= g(x, y) + o(1) + \iint \log(|w - z|^{-1}) \theta_\varepsilon(x - w) \theta_\varepsilon(y - z) dw dz \\ &= \log(1/|x - y|) + g(x, y) + o(1). \end{aligned}$$

Altogether,

$$\text{Cov}(h_\varepsilon(x), h_\varepsilon(y)) = -\log(|x - y| \vee \varepsilon) + g(x, y) + o(1). \quad (3.3)$$

Also, if $|x - y| \leq \varepsilon_0/3$ (else we bound this probability by one), we have

$$\tilde{\mathbb{P}}(G_\varepsilon(x), G_\varepsilon(y)) \leq \tilde{\mathbb{P}}(h_{\varepsilon'}(x) \leq \alpha \log 1/\varepsilon')$$

where

$$\varepsilon' = \varepsilon \vee |x - y|/3. \quad (3.4)$$

Furthermore, by Girsanov, under $\tilde{\mathbb{P}}$ we have that $h'_\varepsilon(x)$ has the same variance as before (therefore $\log 1/\varepsilon' + O(1)$) and a mean given by

$$\text{Cov}_{\mathbb{P}}(h_{\varepsilon'}(x), \gamma h_\varepsilon(x) + \gamma h_\varepsilon(y)) \leq 2\gamma \log 1/\varepsilon' + O(1). \quad (3.5)$$

To see why (3.5) holds, observe that

$$\text{Cov}_{\mathbb{P}}(h_{\varepsilon'}(x), \gamma h_\varepsilon(x) + \gamma h_\varepsilon(y)) \leq \gamma \text{Cov}(h_{\varepsilon'}(x), h_\varepsilon(x)) + \gamma \text{Cov}(h_{\varepsilon'}(x), h_\varepsilon(y)).$$

The first term gives us $\gamma \log(1/\varepsilon')$ by (3.1). For the second term, we only consider the case $|x - y| \geq \varepsilon$ and $\varepsilon' = |x - y|/3$ (as the desired bound in the case $|x - y| \leq \varepsilon$ follows directly from Cauchy-Schwarz). In that case, note that for every $w \in B_{\varepsilon'}(x)$ and every $z \in B_\varepsilon(y)$, we have that $|z - w| \geq 2|x - y| = 6\varepsilon'$ so

$$\iint \theta_{\varepsilon'}(x - w) \theta_\varepsilon(y - z) \log(|w - z|^{-1}) dw dz \leq \log |x - y|^{-1} = \log(1/\varepsilon') + O(1).$$

From this (3.5) follows.

Hence

$$\begin{aligned} \tilde{\mathbb{P}}(h_{\varepsilon'}(x) \leq \alpha \log 1/\varepsilon') &= \mathbb{P}(\mathcal{N}(2\gamma \log(1/\varepsilon'), \log 1/\varepsilon') \leq \alpha \log(1/\varepsilon') + O(1)) \\ &\leq \exp(-\frac{1}{2}(2\gamma - \alpha)^2(\log(1/\varepsilon') + O(1))) = O(1)\varepsilon'^{(2\gamma - \alpha)^2/2}. \end{aligned} \quad (3.6)$$

We deduce

$$\mathbb{E}(J_\varepsilon^2) \leq O(1) \int_{S \times S} |(x - y) \vee \varepsilon|^{(2\gamma - \alpha)^2/2 - \gamma^2} \sigma(dx) \sigma(dy). \quad (3.7)$$

(We will get a better approximation in the next section). Clearly this is bounded if

$$(2\gamma - \alpha)^2/2 - \gamma^2 > -d$$

and since α can be chosen arbitrarily close to γ this is possible if

$$d - \gamma^2/2 > 0 \text{ or } \gamma < \sqrt{2d}. \quad (3.8)$$

This proves the lemma. \square

To finish the proof of Proposition 3.1, observe that $I_\varepsilon = J_\varepsilon + J'_\varepsilon$. We have $\mathbb{E}(J'_\varepsilon) \leq p(\varepsilon_0)$ by Lemma 3.4, and for a fixed ε_0 , J_ε is bounded in L^2 (uniformly in ε). Hence I_ε is uniformly integrable. \square

4 Convergence

As before, since $\mathbb{E}(J'_\varepsilon)$ can be made arbitrarily small by choosing ε_0 sufficiently small, it suffices to show that J_ε converges in probability and in L^1 . In fact we will show that it converges in L^2 , from which convergence will follow. To do this we will show that (J_ε) forms a Cauchy sequence in L^2 . To approach this question, we write

$$\mathbb{E}((J_\varepsilon - J_\delta)^2) = \mathbb{E}(J_\varepsilon^2) + \mathbb{E}(J_\delta^2) - 2\mathbb{E}(J_\varepsilon J_\delta) \quad (4.1)$$

Our basic approach is thus to estimate better than before $\mathbb{E}(J_\varepsilon^2)$ from above and $\mathbb{E}(J_\varepsilon J_\delta)$ from below. Essentially, the idea is that for x, y which are at a small but macroscopic distance, we can identify the limiting distribution of $(h_{\varepsilon'}(x), h_{\varepsilon'}(y))_{\varepsilon' \leq \varepsilon_0}$ under the distribution \mathbb{P} biased by $e^{\bar{h}_\varepsilon(x) + \bar{h}_\delta(y)}$. On the other hand when x, y are closer than that we know from the previous section that the contribution is negligible.

Lemma 4.1. *We have*

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E}(J_\varepsilon^2) \leq \int_{S \times S} e^{\gamma^2 g(x, y)} \frac{1}{|x - y|^{\gamma^2}} g_\alpha(x, y) dx dy$$

where $g_\alpha(x, y)$ is a function depending on α, ε_0 and γ such that the above integral is finite

Proof. Recall that from (3.3) we already know

$$\mathbb{E}(J_\varepsilon^2) = \int_{S^2} \frac{e^{o(1) + \gamma^2 g(x, y)}}{|x - y|^{\gamma^2}} \tilde{\mathbb{P}}(G_\varepsilon(x), G_\varepsilon(y)) dx dy.$$

We simply have to estimate better $\tilde{\mathbb{P}}(G_\varepsilon(x), G_\varepsilon(y))$. We fix $\eta > 0$ arbitrarily small (in particular, η may and will be smaller than ε_0). If $|x - y| \leq \eta$ we use the same bound as in (3.7). The contribution coming from the part $|x - y| \leq \eta$ can thus be bounded by $O(\eta)$. We thus focus on $|x - y| \geq \eta$.

Then observe that for a fixed $\varepsilon' \leq \varepsilon_0$, and for $\varepsilon \rightarrow 0$, and uniformly over $x \in S$,

$$\text{Cov}(h_{\varepsilon'}(x), h_\varepsilon(x)) \rightarrow \int_D K(x, z) \theta_{\varepsilon'}(z - x) dz \quad (4.2)$$

and likewise, uniformly over $x, y \in S$, $|x - y| \geq \eta$, as $\varepsilon \rightarrow 0$ and $\varepsilon' \leq \varepsilon_0$ is fixed,

$$\text{Cov}(h_{\varepsilon'}(x), h_\varepsilon(y)) \rightarrow \int_D K(z, y) \theta_{\varepsilon'}(z - x) dz \quad (4.3)$$

Consequently, by Girsanov, the joint law of the processes $(h_{\varepsilon'}(x), h_{\varepsilon'}(y))_{\varepsilon' \leq \varepsilon_0}$ under $\tilde{\mathbb{P}}$ converges (uniformly in $|x - y| \geq \eta$) to a joint distribution $(\tilde{h}_{\varepsilon'}(x), \tilde{h}_{\varepsilon'}(y))_{\varepsilon' \leq \varepsilon_0}$ whose covariance is unchanged and whose mean is given by the sum of (4.2) and (4.3) times γ . We deduce, uniformly in $|x - y| \geq \eta$, letting $\tilde{G}(x)$ be the event that $\tilde{h}_{\varepsilon'}(x) \leq \alpha \log(1/\varepsilon')$ for all $\varepsilon' \leq \varepsilon_0$,

$$\tilde{\mathbb{P}}(G_\varepsilon(x), G_\varepsilon(y)) \rightarrow g_\alpha(x, y) := \mathbb{P}(\tilde{G}(x), \tilde{G}(y)) \quad (\varepsilon \rightarrow 0). \quad (4.4)$$

Consequently, as $\varepsilon \rightarrow 0$,

$$\int_{S^2; |x-y| \geq \eta} \frac{e^{o(1) + \gamma^2 g(x, y)}}{|x - y|^{\gamma^2}} \tilde{\mathbb{P}}(G_\varepsilon(x), G_\varepsilon(y)) dx dy \rightarrow \int_{S^2; |x-y| \geq \eta} \frac{e^{\gamma^2 g(x, y)}}{|x - y|^{\gamma^2}} g_\alpha(x, y) dx dy. \quad (4.5)$$

Since we already know that the piece of the integral coming from $|x - y| \leq \eta$ contributes $O(\eta)$, it remains to check that the integral on the right hand side of (4.5) remains finite as $\eta \rightarrow 0$. But we have already seen in (3.6) that for $|x - y| \leq \varepsilon_0/3$, $\tilde{\mathbb{P}}(G_\varepsilon(x), G_\varepsilon(y)) \leq |x - y|^{(2\gamma - \alpha)^2/2 - \gamma^2}$; hence this inequality must also hold for $g_\alpha(x, y)$. Hence the result follows as in (3.8). \square

Lemma 4.2. *We have*

$$\liminf_{\varepsilon, \delta \rightarrow 0} \mathbb{E}(J_\varepsilon J_\delta) \geq \int_{S \times S} e^{\gamma^2 g(x, y)} \frac{1}{|x - y|^{\gamma^2}} g_\alpha(x, y) dx dy$$

Proof. In fact, the proof is almost exactly the same as in Lemma 4.1, except that $\tilde{\mathbb{P}}$ is now weighted by $e^{\bar{h}_\varepsilon(x) + \bar{h}_\delta(y)}$ instead of $e^{\bar{h}_\varepsilon(x) + \bar{h}_\varepsilon(y)}$. But this changes nothing to the argument leading up to (4.4) and hence (4.5) still holds. Since we get a lower bound by restricting ourselves to $|x - y| \geq \eta$, we deduce immediately that

$$\liminf_{\varepsilon, \delta \rightarrow 0} \mathbb{E}(J_\varepsilon J_\delta) \geq \int_{S^2; |x - y| \geq \eta} e^{\gamma^2 g(x, y)} \frac{1}{|x - y|^{\gamma^2}} g_\alpha(x, y) dx dy.$$

Since η is arbitrary, the result follows. \square

Proof of convergence in Theorem 1.2. Using (4.1) together with Lemmas 4.1, 4.2, we see that J_ε is a Cauchy sequence in L^2 for any $\varepsilon_0 > 0$. Combining with Lemma 3.4, it therefore follows that I_ε is a Cauchy sequence in L^1 and hence converges in L^1 (and also in probability) to a limit $I = \mu(S)$. \square

Remark 4.3. Note that $\lim_{\varepsilon \rightarrow 0} \mathbb{E}(J_\varepsilon^2)$ depends on the regularisation θ , even though, as we will see next, $\lim_{\varepsilon \rightarrow 0} I_\varepsilon$ does not.

5 Uniqueness of the limit

For the proof of independence of the limit with respect to the regularising kernel θ we may assume without loss of generality that D is bounded.

Lemma 5.1. *We may write $h = \sum_{i=0}^\infty h_i$ where the h_i are independent continuous Gaussian fields, in the sense that for an arbitrary fixed function $f \in L^2(D, dx)$, $\sum_n (h_n, f)$ converges almost surely and the limit agrees with (h, f) almost surely.*

Proof. This is basically the Karhunen-Loeve expansion of h (see [14]). Since h is only a generalised function we do it carefully. By restricting to a smaller domain D' we may assume without loss of generality that $g(x, y)$ is smooth on \bar{D} . Introduce the Fredholm integral operator

$$Tf(x) = \int_D K(x, y) f(y) dy, \quad f \in L^2(D; dy).$$

Note that T is well defined on $L^2(D)$ and maps $L^2(D)$ into continuous functions on \bar{D} by Lebesgue's dominated convergence theorem. Since D is bounded, we deduce that $T : L^2(D) \rightarrow L^2(D)$. Note further that since $K(x, y) = K(y, x)$ by assumption, T is symmetric with respect to the (Lebesgue) inner product on $L^2(D)$. Observe also that since $K(x, y) \in L^2(D \times D)$, we have that T is a compact operator on $L^2(D)$ (follows from equicontinuity and Arzela-Ascoli). By the spectral theorem for compact symmetric operators, we deduce that there exists an orthonormal basis of $L^2(D)$ consisting

of eigenfunctions $\{f_k\}_{k \geq 0}$ of T (Theorem 7 in Appendix D.5 of [4]). Let λ_k denote the corresponding eigenvalue. We have that $\lambda_k \geq 0$ and $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$ by the same theorem. Observe that trivially f_k must be continuous since $Tf_k = \lambda_k f_k$ and $f_k \in L^2$ so Tf_k is continuous.

We observe that we can write

$$K(x, y) = \sum_{k=0}^{\infty} \lambda_k f_k(x) f_k(y), \text{ a.e.}$$

Indeed for an arbitrary $f \in L^2$,

$$Tf(x) = \sum_k \lambda_k (f, f_k) f_k(x) = \int \left(\sum_k \lambda_k f_k(x) f_k(y) \right) f(y) dy$$

which for a fixed x identifies $K(x, \cdot)$ as an L^2 function.

Now consider our field h . Observe that $\xi_k = (h, f_k)/\sqrt{\lambda_k}$ must be i.i.d. standard Gaussian random variables. (Formally, this is the same as saying that $h(x) = \sum_k \sqrt{\lambda_k} \xi_k f_k(x)$.) Let us define $h_k(x) = \sqrt{\lambda_k} \xi_k f_k(x)$, which is an a.s. continuous function and in $L^2(D)$. Observe that for an arbitrary test function f , $\sum_{k=0}^n (h_k, f)$ defines a martingale in the filtration generated by (ξ_1, \dots) and has a variance bounded by $\sum_{k=0}^{\infty} \lambda_k f_k(x) f_k(y) = \text{Var}((h, f))$. Hence the martingale converges a.s. and in $L^2(\mathbb{P})$. Moreover the same calculation shows that in fact $(h, f) - \sum_{k=0}^N h_k f_k$ converges to 0 a.s. in $L^2(\mathbb{P})$. Hence the limit of the martingale agrees with (h, f) . \square

Now define $h^n(z) = \sum_{k=0}^n h_k(z)$ and set

$$\mu^n(S) = \int_S \exp \left(\gamma h^n(z) - \frac{\gamma^2}{2} \text{Var}(h^n(z)) \right) dz.$$

Then $\mu^n(S)$ is a positive martingale with respect to the filtration $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ so converges to a limit which we will call $\mu'(S)$.

Lemma 5.2 (Uniform integrability of μ_ε implies uniform integrability of μ^n). *The sequence of random variables $(\mu^n(S))_{n \geq 0}$ is also uniformly integrable.*

Proof. Observe that

$$\mathbb{E}(\mu_\varepsilon(S) | \mathcal{F}_n) = \mu_\varepsilon^n(S)$$

where μ_ε^n is defined as μ^n except with h^n replaced by its regularisation $h_\varepsilon^n = h^n \star \theta_\varepsilon$. [This follows from writing $h = h^n + X$ where X is independent from h^n .] When n is finite and $\varepsilon \rightarrow 0$ there is no problem in making the right hand side converge to $\mu^n(S)$ by continuity of h_n . Hence the left hand side also converges to some limit as $\varepsilon \rightarrow 0$, and we have

$$\mu^n(S) = \lim_{\varepsilon \rightarrow 0} \mathbb{E}(\mu_\varepsilon(S) | \mathcal{F}_n). \tag{5.1}$$

Recall that $I_\varepsilon = \mu_\varepsilon(S)$ is uniformly integrable by Proposition 3.1. Hence (by the De La Vallée Poussin theorem) there exists a convex increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{x \rightarrow \infty} \frac{\phi(x)}{x} = \infty$$

and $\mathbb{E}(\phi(I_\varepsilon)) \leq C$ is bounded. Hence (by continuity of ϕ and Jensen's inequality),

$$\phi(\mu^n(S)) = \lim_{\varepsilon \rightarrow 0} \phi(\mathbb{E}(I_\varepsilon | \mathcal{F}_n)) \leq \liminf_{\varepsilon \rightarrow 0} \mathbb{E}(\phi(I_\varepsilon) | \mathcal{F}_n).$$

Taking expectations, and using Fatou's lemma,

$$\mathbb{E}(\phi(\mu^n(S))) \leq \liminf_{\varepsilon \rightarrow 0} \mathbb{E}(\phi(I_\varepsilon)) \leq C.$$

Hence $\mu^n(S)$ is uniformly integrable. □

Proof of Theorem 1.2. It suffices to show that $\mu(S) = \mu'(S)$. This will show that $\mu(S)$ does not depend on the regularisation kernel.

From (5.1) and Fatou's lemma, we see that

$$\mu^n(S) \geq \mathbb{E}(\mu(S) | \mathcal{F}_n).$$

Letting $n \rightarrow \infty$, and using the martingale convergence theorem,

$$\mu'(S) \geq \mathbb{E}(\mu(S) | \mathcal{F}_\infty) = \mu(S)$$

where the equality holds since $\mu(S)$ is measurable with respect to \mathcal{F}_∞ (indeed, for a fixed $\varepsilon > 0$ and $x \in D$, $h_\varepsilon(x) = h \star \theta_\varepsilon(x)$ is measurable with respect to \mathcal{F}_∞ as an almost sure limit of \mathcal{F}_∞ -measurable quantities, and recall that $h_\varepsilon(x)$ is a.s. continuous in x . So $\mu_\varepsilon(S)$ is also \mathcal{F}_∞ -measurable). But by Lemma 5.2, $\mu_n(S)$ is uniformly integrable so $\mathbb{E}(\mu'(S)) = \lim_{n \rightarrow \infty} \mathbb{E}(\mu^n(S)) = 1$. Likewise, by Proposition 3.1, $\mathbb{E}(\mu(S)) = 1$ as well. Consequently, $\mu(S) = \mu'(S)$ and we are done. □

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